Towards empirical assessments of controlled cointegrated models

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Abstract

This paper explores control theory and stabilisation policy within the framework of a cointegrated vector autoregressive (VAR) model, adopting the perspective of an applied econometrician.We demonstrate that a *new* process derived from control theory should be regarded as a series of observables. This process can be viewed as driven by a vector autoregressive moving-average (VARMA) model, which, in turn, can be interpreted through a structural VAR framework. This approach enables the econometrician to identify and evaluate policy interventions. We also introduce a data-driven procedure for classifying intermediate and final policy targets within the model. The practicality and effectiveness of this procedure are demonstrated through a counterfactual policy analysis of New Zealand's monetary policy data.

JEL classification codes: C32, C54, E52

Keywords: Cointegrated vector autoregressive model; Structural vector autoregressive model; Control theory; Policy targets; Counterfactual analysis; Monetary policy.

1 Introduction

This paper explores control theory within the framework of a cointegrated vector autoregressive (CVAR) model, viewed from the standpoint of an econometrician conducting counterfactual analyses or assessing, *ex post*, the presence and effectiveness of stabilisation policies. We build extensively on Johansen and Juselius (2001), hereafter referred to as JJ.

JJ introduced control theory in a CVAR framework to derive policies that aim at stationarizing a combination of target variables. A number of empirical studies have since used

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their theory in various contexts: for macroeconomic and financial data, see Christensen and Nielsen (2009), Carlucci and Montaruli (2014), Boug *et al.* (2024), and Castle and Kurita (2024), among others; for climate data, see Chevillon and Kurita (2024). The article by Johansen and Juselius in this special issue presents a revised and extended version of their previous work, JJ, delineating the core concepts of their method. Note that in this study, we refer to their *original* 2001 paper as JJ, rather than the article published in this issue, but we point out in our analysis where the two papers markedly differ in a manner that matters to us.

This paper adopts the perspective of an applied econometrician aiming to assess policies within the class of models considered by JJ. While most previous studies have focused on simulating *counterfactual* retroactive or prospective policies, we examine here how an econometrician can estimate and evaluate a policy *ex post*, after its implementation. This requires identifying the observables and determining how to estimate the parameters of a model for the generated observables. Based on our previous investigations, we find that the framework and policy narrative presented by JJ in their *original* paper brings challenges when attempting to identify the observed data and, consequently, in estimating a CVAR model for the underlying data-generating process. However, building on Rambachan and Shephard (2021), we propose an alternative interpretation for the implementation of control policy along JJ's mathematical results. We demonstrate that data observed *after* a policy has been implemented can be represented as a vector autoregressive moving-average process, or a VARMA$(p, 1)$ process, which can be given a structural vector equilibrium correction (SVEC) interpretation, where the lagged MA(1) innovation constitutes the *policy* shock. This enables us to examine whether an applied econometrician can indeed identify the policy *ex post*.

This allows us to extend JJ's analysis and explore in greater depth some of the mechanisms governing control policy, with the aim of delineating how the policy maker can implement their policies and the econometrician can assess their success. For this we focus on the situation where they cannot directly control their desired target through their instrument, but must rely on market forces through an intermediate target. This is a situation that was already studied by JJ. While their definitions of intermediate and final targets are clear and unproblematic in theoretical contexts, challenges arise when these concepts are applied empirically. Specifically, distinguishing between intermediate and final policy targets among a set of candidate variables is complex in practice, particularly when analysing the data of multiple candidate variables in a CVAR system. Although insights from economic theory are valuable, they are often insufficient to justify an *a priori* distinction in most cases. One of the aims of this paper is to propose a procedure for identifying feasible intermediate and final policy targets in empirical contexts. An empirical illustration of the proposed procedure is provided by modeling a New Zealand macroeconomic dataset.

New Zealand was chosen for the following reasons: (i) it is a front-runner in inflation targeting policy, having been the first country to formally adopt the policy in the early 1990s and is well-known for its successful implementation over the past three decades; and (*ii*) the Reserve Bank of New Zealand (RBNZ) has published long-term time series data on inflation expectations. While the suggested procedure can be applied to time series from other economies, New Zealand's data are particularly suitable for demonstrating the usefulness of our procedure in the context of policy simulation analysis. For a preceding empirical illustration of applying a cointegrated method to New Zealand's time series data, see Choo and Kurita (2016), *inter alia*.

The rest of this paper is organised into four sections. Section 2 briefly reviews control theory within a CVAR system and then considers issues faced by an econometrician working with the data generated after a policy has been implemented. Section 3 addresses the issue of identifying intermediate and final policy targets and considers an empirical procedure. Section 4 provides an empirical illustration of the procedure. Finally, Section 5 presents concluding remarks. All econometric analyses in this paper were conducted using *Cats* (Doornik and Juselius, 2023), *Ox* (Doornik, 2023) and *PcGive* (Doornik and Hendry, 2023).

2 Inference in controlled cointegrated systems

This section revisits control theory within the context of a CVAR model through the perspective of an econometrician willing to estimate a model and perform counterfactual analyses. We begin by reviewing JJ's control theory and then discuss various methodological issues related to counterfactual policy analysis using a CVAR system.

2.1 CVAR-based control theory

We start by providing a brief review of control theory in a CVAR model for I(1) nonstationary time series data; for further details of the model, refer to Johansen (1988, 1996), Juselius (2006) and Hunter *et al.* (2017). Let X_t be a p-dimensional vector of time series which is represented as the following trend-restricted CVAR(k) model conditional on $X_{-k+1}, \ldots, X_0$:

$$\Delta X_t = \alpha \left(\beta', \rho\right) \begin{pmatrix} X_{t-1} \\ t \end{pmatrix} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \tau + \varepsilon_t, \quad \text{for} \quad t = 1, ..., T, \tag{1}$$

where ε_t is a martingale-difference sequence with a positive definite variance matrix $\Omega \in \mathbf{R}^{p\times p}$, a process satisfying a class of assumptions provided by Kurita and Nielsen (2019). The parameters of (1) are defined as $\alpha, \beta \in \mathbf{R}^{p\times r}$ for $r < p$, $\Gamma_i \in \mathbf{R}^{p\times p}$, $\rho \in \mathbf{R}^r$ and $\tau \in \mathbf{R}^p$. The parameters α (adjustment or loading vectors) and β (cointegrating vectors) are assumed to be of full rank r (the cointegration rank). Let their orthogonal complements $\alpha_\perp, \beta_\perp \in \mathbf{R}^{p\times(p-r)}$ of full rank $p - r$, so that the equality $\alpha'_\perp \alpha = \beta'_\perp \beta = 0$ holds along with the non-singularity of the two matrices $(\alpha, \alpha_\perp)$ and $(\beta, \beta_\perp)$. In order to justify I(1) CVAR analysis rather than I(2) or higher order degrees of integration, we assume that $\alpha_\perp' \Gamma \beta_\perp$ is of full rank $p - r$ for $\Gamma = I_p - \sum_{i=1}^{k-1} \Gamma_i$. In the development of control theory in the next subsection, it is also essential to introduce here $C = \beta_\perp (\alpha_\perp' \Gamma \beta_\perp)^{-1} \alpha_\perp'$, known as the impact matrix in the Granger-Johansen representation.

The control theory developed by JJ considers a policy aiming at stabilising a subset of the system variables, or a linear combination thereof, so they become stationary around a specified mean. We simplify here the model (1) into a constant-restricted model with $k = 1$ to make the required argument straightforward (but provide a proof for the general case in the Appendix):

$$\Delta X_t = \alpha(\beta' X_{t-1} - \mu) + \varepsilon_t, \quad \text{for} \quad t = 1, ..., T, \tag{2}$$

for $\mu \in \mathbf{R}^r$. Equation (2) provides a basis for a review of the theory.

We now introduce two policy matrices $a, b \in \mathbf{R}^{p \times m}$ for $m + r < p$. The matrix a is associated with the selection of policy instruments, $a' X_t$, while the matrix b pertains to the selection of policy targets, $b' X_t$. The aim of the policy is to stabilise $b' X_t$ using $a' X_t$, which means making $b' X_t$ stationary with mean b^* – the policy target level – through the use of $a' X_t$. Achieving this requires a t-timed contemporaneous policy intervention, represented by $\kappa' X_t - \kappa^*$ for $\kappa \in \mathbf{R}^{p \times m}$ and $\kappa^* \in \mathbf{R}^m$. Policy implementation replaces X_t with X_t^{ctr}, dubbed the '*controlled*' process, which is defined as

$$X_t^{ctr} = X_t + \overline{a} \left(\kappa' X_t - \kappa^* \right),$$

for $\overline{a} = a(a'a)^{-1}$.

Given X_t^{ctr}, assuming market dynamics are not modified by the intervention, equation (2) generates a '*new*' series X_t^{new}, with the requirement that $b' X_t^{new}$ is stationary with mean b^*. The overall process is hence two-staged, so that from a policy inception at date t_0, and letting $\nu : x \to \overline{a} \left(\kappa' x - \kappa^* \right)$,

$$X_{t_0} \to \underbrace{X_{t_0}^{ctr} = X_{t_0} + \nu \left(X_{t_0} \right)}_{\text{(Policy)}} \to \underbrace{X_{t_0+1}^{new} = \left(I_p + \alpha \beta' \right) X_{t_0}^{ctr} - \alpha' \mu + \varepsilon_{t_0+1}}_{\text{(Ecosystem)}}$$

$$\to \underbrace{X_{t_0+1}^{ctr} = X_{t_0+1}^{new} + \nu \left(X_{t_0+1}^{new} \right)}_{\text{(Policy)}} \to ....$$

To determine the parameters of the policy rule that achieve stabilisation, we see that equation (2) implies that the long run response of the system satisfies

$$X_\infty \equiv \lim_{h \to \infty} \mathrm{E}(X_{t_0+h} \,| X_{t_0}) = C X_{t_0} + \alpha(\beta' \alpha)^{-1} \mu.$$

The long-run response of the economic process to policy introduction is therefore, in the directions defined by the policy target b, $\lim_{h \to \infty} \mathrm{E} \left(b' X_{t_0+h}^{new} \,\middle|\, X_{t_0}^{ctr} \right) = b' \left\{ C X_{t_0}^{ctr} + \alpha(\beta' \alpha)^{-1} \mu \right\}$.

Hence $b'X_t^{new}$ having been stationarised by the intervention around b^* means that, in terms of the original variable,

$$b^* = b'\left\{C\left[X_t + \overline{a}\left(\kappa' X_t - \kappa^*\right)\right] + \alpha(\beta'\alpha)^{-1}\mu\right\}.$$

Hence, if $\det(b'Ca) \neq 0$ holds, a policy rule that satisfies the requirements can be written as

$$\kappa' X_t - \kappa^* = -(b'C\overline{a})^{-1}\left[b'CX_t - b^* + b'\alpha(\beta'\alpha)^{-1}\mu\right],$$

so that $\kappa' = -(b'C\overline{a})^{-1}b'C$ and $\kappa^* = -(b'C\overline{a})^{-1}\left[b^* - b'\alpha(\beta'\alpha)^{-1}\mu\right]$ are solutions. We thus refer hereafter to $\det(b'Ca) \neq 0$ as the controllability condition.

The identity $C = I_p - \alpha\left(\beta'\alpha\right)^{-1}\beta'$ then leads to the following important equation

$$\kappa' X_t - \kappa^* = (b'C\overline{a})^{-1}\left[b'\alpha(\beta'\alpha)^{-1}(\beta' X_t - \mu) - (b'X_t - b^*)\right],$$

i.e., the policy rule constitutes of weighted average of two forms of disequilibria, given respectively by $\beta' X_t - \mu$, a vector of deviations from the long-run relationships, and $b'X_t - b^*$, the discrepancy between the actual and desired targets.[1]

As the policy needs to be implemented every period, $X_t^{ctr} = X_t^{new} + \overline{a}\left(\kappa' X_t^{new} - \kappa^*\right)$ for all $t > t_0$ and JJ derive the corresponding *new* dynamics:

$$\Delta X_{t+1}^{new} = \left[\alpha, \left(I_p + \alpha\beta'\right)\overline{a}\right]\begin{pmatrix} \beta' X_t^{new} - \mu \\ \kappa' X_t^{new} - \kappa^* \end{pmatrix} + \varepsilon_{t+1}. \tag{3}$$

The *new* system is characterised by an additional cointegration relation corresponding to the implemented policy.

2.2 The econometrician's problem

We now consider JJ's analysis from the perspective of an econometrician aiming to identify policy intervention and conduct counterfactual analyses. We study the elements this econometrician must consider in turn.

2.2.1 Timing and observables

The principle of JJ's approach to policy is that the control rule is applied at each point in time, thereby defining – implicitly or explicitly – two processes (X_t^{new}, X_t^{ctr}), which accord to a specific timing. Let us repeat it here as it is important for our discussion.

[1] Note that in the recent version of their theory presented in this special issue, JJ no longer refer explicitly to the *controlled* process, but it can still be defined as the result of the authority's intervention, so the *new* – stablized – process requires that market dynamics, in the form of equation (2), remain unaffected by the transform $X_t \to X_t^{ctr} = X_t + \nu\left(X_t\right)$. In this context, it is still reasonable to ask what the observables are.

1. At the beginning of period t, the policy-making authority observes the process that is generated by market forces. We denote it by X_t^{new} for simplicity (at policy inception, $t = t_0$, we let $X_{t_0}^{new} = X_{t_0}$). The authority chooses an intervention $\nu(\cdot)$ that modifies X_t^{new} and generates the *controlled* process:

$$X_t^{ctr} \equiv X_t^{new} + \nu\left(X_t^{new}\right).$$

2. The market at time $t+1$ generates the next value X_{t+1}^{new} of the process:

$$X_{t+1}^{new} = \left(I_p + \alpha\beta'\right) X_t^{ctr} - \alpha'\mu + \varepsilon_{t+1}.$$

3. The authority intervenes again and sets $X_{t+1}^{ctr} = X_{t+1}^{new} + \nu\left(X_{t+1}^{new}\right)$, and so on.

In the narrative of JJ, the intervention is contemporaneous so X_t^{new} is immediately transformed into X_t^{ctr}, and this is the process that market forces then work with to generate the next period's observations. Given that the policy is implemented at every period (between inception and termination), the process observed by the econometrician must be X_{t+1}^{ctr}. The *new* process X_{t+1}^{new} corresponds to a latent state, as it is immediately transformed by the authority and never actually holds.

An alternative timing for decisions is possible, which may render both processes observables. This would require introducing subperiods at t. For instance, if we consider the Federal Funds rate as a policy instrument, then markets and the econometrician observe X_t^{new} at the beginning of the period (a month or a quarter). The decision or intervention is then made at the FOMC meeting during the period, at which point X_t^{ctr} is generated and becomes the *new* value for the entire vector of t-timed observables. The outcome for the next period becomes available in its first half — X_{t+1}^{new} first, followed by X_{t+1}^{ctr} later in the period. One issue with this interpretation is that both X_t^{new} and X_t^{ctr} correspond to the same set of variables. If the Federal Funds rate is part of X_t^{new}, the observed value is contingent on the policy being followed; it cannot be set by both markets and authorities directly. Otherwise, we are dealing with two distinct concepts that must be represented by separate variables. The only solution would be to assume that the policy instrument is set solely by the authority, so that Ω, the variance-covariance of ε_t, is a singular matrix, with zero variance in the direction of a, *i.e.*,

$$\text{Var}\left(a'\varepsilon_t\right) = 0.$$

In the example of the Federal Funds rate, the variance of its innovations in the CVAR 'market' representation (absent policy interventions) must therefore be zero.

This alternative timing does not align with the system's assumptions in the absence of the policy, so it cannot hold. Consequently, the process observable by the econometrician should be *a priori* X_t^{ctr}, while X_t^{new} should be latent.

2.2.2 What's wrong with the *controlled* process?

Let us consider the consequences of the discussion above, where the policy generates a unique set of observables, X_t^{ctr}. While the theory has established that X_t^{new} exhibits an increased rank of cointegration, as in expression (3), and that $b'X_t^{new}$ is stationary about b^*, the dynamics of X_t^{ctr} is atypical. Indeed, the natural control rule derived by JJ is such that $\kappa'\alpha = 0$, and, $I_m + \kappa'\overline{a} = 0$. Notice in this case that, unlike the latent X_t^{new}, whose rank of cointegration is $r+m$, the observable X_t^{ctr} maintains a cointegration rank of r, with modified reduced-form errors:

$$\Delta X_{t+1}^{ctr} = \alpha \left(\beta' X_t^{ctr} - \mu\right) + \left(I_p + \overline{a}\kappa'\right) \varepsilon_{t+1}, \tag{4}$$

which is compatible with

$$\kappa' x_{t+1}^{ctr} = \kappa^*.$$

See the proof in the Appendix. Thus, for the *controlled* variable X_t^{ctr}, the cointegration properties remain unchanged compared to the process without control. However, the innovations to X_{t+1}^{ctr} in (4) exhibit a singular variance-covariance structure due to

$$\det\left[\left(I_p + \overline{a}\kappa'\right)\Omega\left(I_p + \overline{a}\kappa'\right)'\right] = 0.$$

See the Appendix for a proof. This reduced rank of the variance-covariance of the innovations implies that the *controlled* process exhibits peculiar dynamics. This is clear from the control policy which ensures that at all times

$$b'\left[CX_t^{ctr} + \alpha\left(\beta'\alpha\right)^{-1}\mu\right] = b^*,$$

i.e. a linear combination of X_t^{ctr} remains constant at every period. Rewriting the above, we see that

$$b'X_t^{ctr} - b^* = b'\alpha\left(\beta'\alpha\right)^{-1}\left(\beta' X_t^{ctr} - \mu\right).$$

Hence $\beta' X_t^{ctr} - \mu \sim \mathsf{I}(0)$ implies that $b'X_t^{ctr} - b^* \sim \mathsf{I}(0)$ and the two are collinear. The increased rank of cointegration in X_t^{new} is only implicitly present in X_t^{ctr} since the second cointegration relation is directly proportional to the first, for this process. The reduced rank of the covariance of the innovation for X_t^{ctr} implies that, when this constitutes the observable process, cointegration analysis will lead misleading results for the econometrician as we show next by simulation.

2.3 Monte Carlo evidence

We document the extent to which degenerate dynamics for X_t^{ctr} impair inference on the cointegrating rank and, consequently, on the policy evidence. For this analysis, we consider a scenario in which the authority implements a univariate policy ($m = 1$). To maintain some plausibility of policy control, we assume that the parameters of the CVAR model are estimated using reduced-rank regression, rather than the true parameters, when forming the policy. We then adopt the perspective of an econometrician with no prior knowledge of the underlying cointegrating rank and evaluate how frequently the econometrician correctly identifies the rank of cointegration across the *original*, and generated *controlled* and *new* processes. Regarding the data generation process (DGP) we let it follow a CVAR(1) but modify system dimensions and cointegration ranks. See the Appendix for further details of DGPs employed in the Monte Carlo simulation. Using 10,000 Monte Carlo replications we first report the rejection frequencies of the null of a given rank r of cointegration, using the usual cointegration trace statistics at a nominal size of 5%.

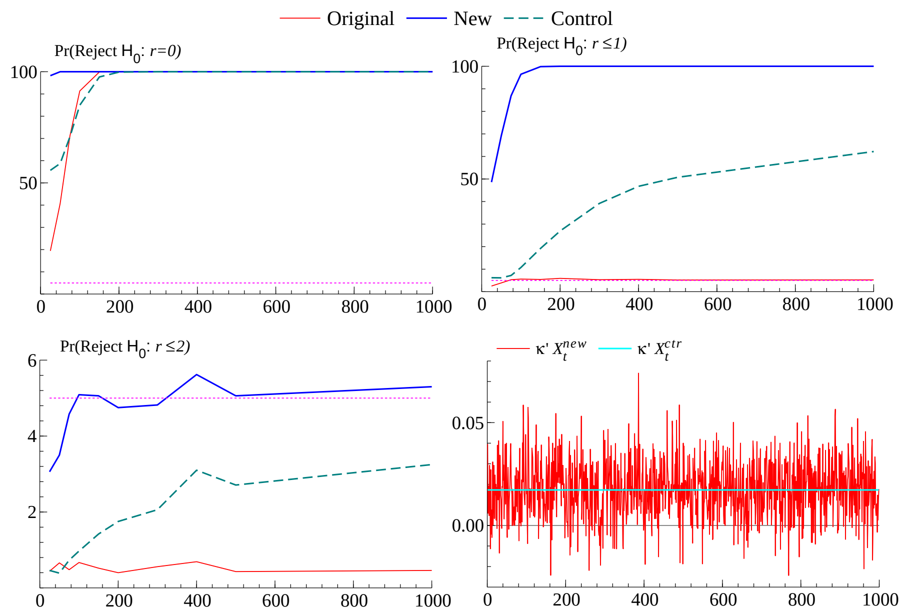

Figure 1: Panels (a)-(c): Rejection frequencies, at the 5% nominal level, of the trace test of a cointegration rank r in a 3-variate system as a function of the sample size (horizontal axis). Each panel corresponds to a different hypothesized value of r when the truth if 1 for the original data, and 2 for the *new* data. Distributions are obtained by simulation over 10,000 replications. Panel (d) on the bottom right presents one realization of the processes over a sample of dimension 1000.

Figure 1 records such rejection frequencies for a range of null hypotheses concerning a three-variate system. In the DGP, the rank of cointegration is $r = 1$ for X_t (denoted Original, in the figure), and $r = 2$ for X_t^{new} (New, in the figure) as policy control increases the cointegrating rank. Over moderate samples of sizes greater than 200 observations, rejection frequencies are close to the nominal values under the null ($r \leq 1$ for X_t and $r \leq 2$ for X_t^{new}). The power is also high over the same sample size. By contrast, we observe massive distortions about X_t^{ctr} (Control, in the figure). Because of its DGP's innovation covariance matrix rank degeneracy, the test statistic rejects on average 50% of the time for the null of 1

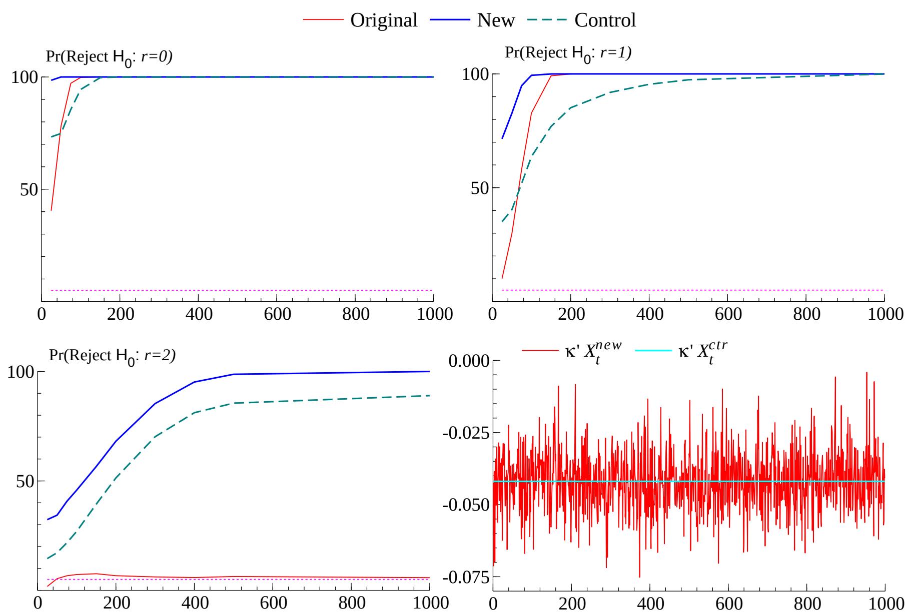

Figure 2: Panels (a)-(c): Rejection frequencies, at the 5% nominal level, of the trace test of a cointegration rank r in a 3-variate system as a function of the sample size (horizontal axis). Each panel corresponds to a different hypothesized value of r when the truth if 2 for the original data, and 3 for the *new* data. Distributions are obtained by simulation over 10,000 replications. Panel (d) on the bottom right presents one realization of the processes over a sample of dimension 1000.

cointegrating relation, and 2 to 3% for the null $r \leq 2$. Treating the '*controlled* process' as the observables to test for the presence of an increased rank of cointegration would therefore be misleading, with low power at $r \leq 1$ and conservative size at $r \leq 2$. Testing the correct null that $r \leq 1$ would lead to massive overrejection, owing to the singular innovation covariance matrix. To explore the issues further, we consider in Figure 2 the situation of an initial rank of cointegration $r = 2$ so the policy renders all processes stationary. In this situation, inference on the *controlled* process is more similar to that of the *new* process, except that the probability not to reject $r = 2$ is higher by about 15%. In both figures, the bottom right panel presents $\kappa' X_t^{new}$ and $\kappa' X_t^{ctr}$ and we see that the former appears stationary and the latter constant.

Given that inference on the rank of cointegration follows a sequential testing procedure, we complement the previous results by a simulation where we record the frequency with which a specific rank of cointegration r_0 is selected, such that the procedure rejects $r = 0, ..., r_0 - 1$ and does not reject r_0 at the 5% nominal size (if $r_0 < p$ or this latter null is not tested). These results are presented in Figure 3 for the two DGPs considered previously, with $r = 1$ and 2. The figure shows, on the left hand side where $(p, r) = (3, 1)$, that for the original and *new* processes, the selection procedure achieves rates close to 95% of the correct cointegration

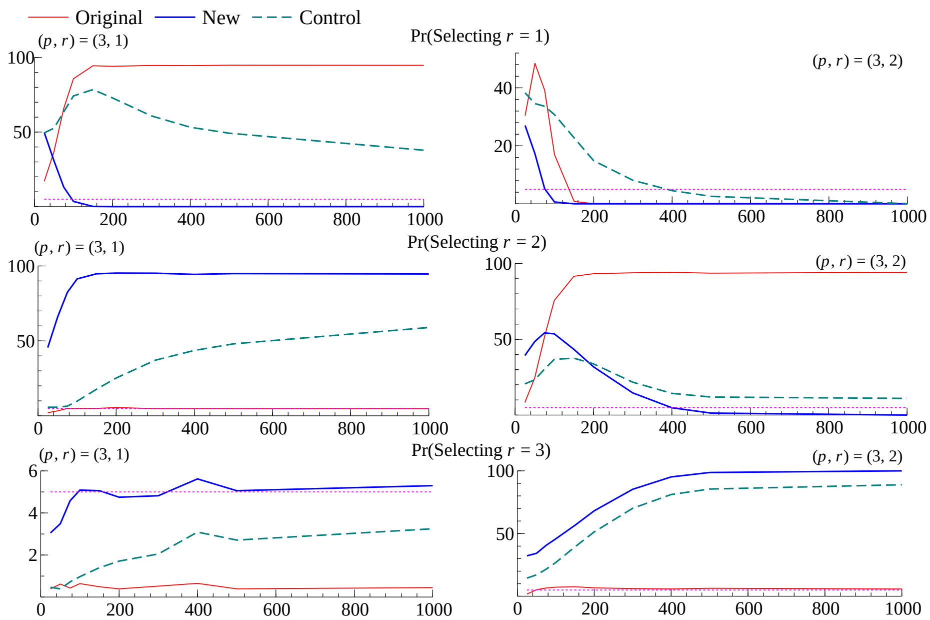

Figure 3: Probability to select a specific rank of cointegration using the sequential testing procedure based on the trace test at the 5% asymptotic nominal size. Each row corresponds to the selection of a different rank r. The cointegration rank of the original data is 1 in the left colum, 2 in the right column. Horizontal axes record the sample size.

rank. Yet for the *controlled* process, this rate is about 50% for both $r_0 = 1$ or 2. On the right-hand side panels, where $(p, r) = (3, 2)$ we see that the procedures works well for the original process and for the *new* process. The *controlled* version selects $r = 3$ with a higher frequency than that of selecting $r = 2$ on the left column, but still less so than the *new* process.

In order to shed more light on the reasons for the results above, Figure 4 records the distribution of the estimators of test statistics over a sample of $T = 1{,}000$ observations, together with that under the limiting distribution under the null. We used 10,000 Monte Carlo replications to simulate distributions. We report only the situation $(p, r) = (3, 1)$ but similar results hold for other values. We see that the difference between inference on the *new* and *controlled* processes lies essentially in that, while the trace statistic for the null $r \leq 1$ rejects strongly for the *new* process, it is, for its *controlled* counterpart, correctly centered on the limiting distribution but with very large variability caused by the innovation variance singularity. This explains the high rejection rate we established before.

We present in the Appendix similar results for different parameter settings but with the same number of observations and replications. All of these results indicate that inference based on the *controlled* process is unreliable. This may seem to pose a problem for the econometrician who wishes to perform inference in the policy setting considered above.

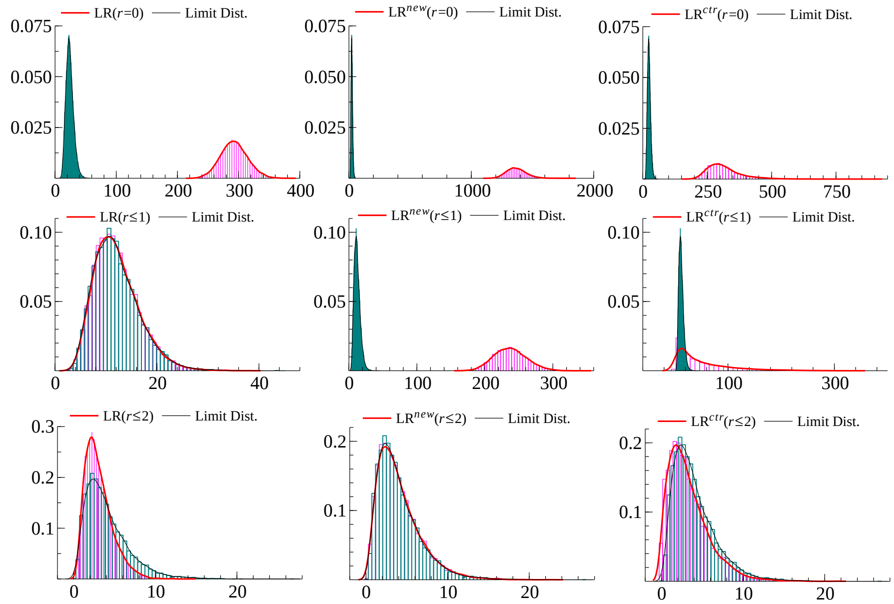

Figure 4: Distribution of cointegration test statistics (each row denotes a different null for r) at a sample of $T = 1,000$ observations in a 3-variate setting. The left column corresponds to the original data, the central one to the new data, and the column on the right to the controlled data.

2.4 A new understanding

Fortunately, while the previous analysis may suggest that assessing policy empirically could be challenging within the JJ framework, we believe it is possible to put it to the data. In fact, we find that we can circumvent the need to handle *controlled* data. Our approach hinges on a reinterpretation of the policy narrative, informed by recent research on the topic.

The route we follow consists of a reinterpretation of the timing of the policy, using a framework delineated by Rambachan and Shephard (2021) and we borrow their explanations. In their approach to policy, at each period $t \geq 1$, the unobserved unit X_t receives a random assignment W_t and we observe an outcome $X_t^{new}(W_t)$. The "potential outcome" process at time t, for any deterministic sequence $\{w_s\}$, is $X_t^{new}\left(\{w_s\}_{s\geq 1}\right)$. Under the assumption of *Non-anticipating Potential Outcomes*, for each $t \geq 1$ and all deterministic sequences $\{w_t\}_{t\geq 1}$, $\{w'_t\}_{t\geq 1}$, the potential outcomes do not depend on future realisations, i.e.,

$$X_t^{new}\left(w_{1:t}, \{w_s\}_{s\geq t+1}\right) \overset{a.s.}{=} X_t^{new}\left(w_{1:t}, \{w'_s\}_{s\geq t+1}\right).$$

Rambachan and Shephard make the link with the macroeconomic literature on impulse response functions (IRF), defined in the context of Structural VARs as (Sims *et al.*, 1982) for $h \geq 1$ as

$$\mathsf{IRF}_{k,t,h}\left(w_k, w'_k\right) \equiv \mathsf{E}\left[Y_{t+h}\left(W_{k,t}\right) \middle| W_{k,t} = w_k\right] - \mathsf{E}\left[Y_{t+h}\left(W_{k,t}\right) \middle| W_{k,t} = w'_k\right].$$

Rambachan and Shephard (2021) show that the IRF can be given a causal meaning, coinciding with the Average Treatment Effect $\mathsf{E}\left[Y_{t+h}\left(w_k\right)-Y_{t+h}\left(w_k'\right)\right]$ under some orthogonality conditions that are satisfied when the assignment constitutes a *shock*, which they define as satisfying $W_{k,t} \perp \left(W_{1:t-1}, W_{k',t}, W_{t+1:t+h}, \left\{X_{t+h}^{new}\left(w_{1:t+h}\right)\right\}\right)$. From their definitions, the assignment W_t corresponding to the control policy is zero when the DGP for X_t^{new} coincides with that of X_t. We can therefore define the assignment as the control in JJ:

$$W_{t+1}=\pi\nu\left(X_t^{new}\right), \tag{5}$$

for some matrix π to be defined. In our context where the policy is implemented at every period, Rambachan and Shephard (2021) define the *impulse causal effect at horizon $h \geq 1$* as the difference between X_{t+h}^{new} and the *counterfactual* X_{t+h}^{*new} that obtains when the only change is that the policy is not implemented at time t (so $W_t^*=\pi\nu\left(X_{t-1}^{*new}\right)=0$ under the counterfactual – this is the only difference in assignments between X_{t+h}^{new} and X_{t+h}^{*new}). In their words, the impulse causal effect measures the *ceteris paribus* causal effect – of intervening to switch the time-t assignment from 0 to W_t – on the h-period ahead outcomes, holding all else fixed along the assignment process. Since X_t is non-stationary in JJ, the impulse causal effect and its unconditional expectation, the Average Treatment Effect, may vary with time. Yet, we notice that the policy intervention, W_{t+1} in (5) does not constitute a contemporaneous *shock* in the Ramey (2016) or Rambachan and Shephard (2021, Theorem 2) sense, since W_{t+1} is not unanticipated from, or uncorrelated with, lagged endogenous variables, in fact it might a priori be persistent (though stationary under the assumption of controllability). In practice, JJ, Theorem 7, show there exists a linear policy rule which ensures that W_{t+1} can be expressed as a function of the lagged shocks to the unperturbed system and can be made *iid* – even when the original DGP is a VAR(k). In the context of the VAR(1), following on the implementation of the policy, the DGP writes

$$X_{t+1}^{new}=-\left[\alpha\mu+\left(I_p+\alpha\beta'\right)\bar{a}\kappa^*\right]+\left(I_p+\alpha\beta'\right)\left(I_p+\bar{a}\kappa'\right)X_t^{new}+\varepsilon_{t+1},$$

so $\kappa' X_{t+1}^{new}=-\kappa'\alpha\mu+\kappa'\left(I_p+\alpha\beta'\right)\left[\left(I_p+\bar{a}\kappa'\right)X_t^{new}-\bar{a}\kappa^*\right]+\kappa'\varepsilon_{t+1}$. Under policy assumptions $\kappa'\alpha=0$ and $I_m+\kappa'\bar{a}=0$. The previous expression then simplifies as

$$\kappa' X_{t+1}^{new}=\kappa'\left[\left(I_p+\bar{a}\kappa'\right)X_t^{new}-\bar{a}\kappa^*\right]+\kappa'\varepsilon_{t+1}=\kappa^*+\kappa'\varepsilon_{t+1},$$

i.e., setting $\pi=\left(I_p+\alpha\beta'\right)$, we obtain $W_{t+1}=\pi\nu\left(X_t^{new}\right)=\left(I_p+\alpha\beta'\right)\bar{a}\kappa'\varepsilon_t$.

Hence, the DGP under the new policy – the process for the potential outcome – becomes

$$\begin{aligned}\Delta X_{t+1}^{new}&=\alpha\left(\beta' X_t^{new}-\mu\right)+\left(I_p+\alpha\beta'\right)\bar{a}\kappa'\varepsilon_t+\varepsilon_{t+1}, \\ &=\alpha\left(\beta' X_t^{new}-\mu\right)+W_{t+1}+\varepsilon_{t+1},\end{aligned} \tag{6}$$

i.e., a VARMA$(1,1)$ model. Using the results in Theorem 8 of JJ, we can show the same result for a VAR(k) model that becomes a VARMA$(k,1)$ model under the policy. Hence, since $\varepsilon_t \perp \varepsilon_{t+1}$ holds, an alternative SVEC representation is feasible:

$$\Delta X_{t+1}^{new} = \alpha \left(\beta' X_t^{new} - \mu\right) + B \underline{\varepsilon}_{t+1}, \tag{7}$$

where $\underline{\varepsilon}_{t+1} = \left(\varepsilon_{t+1}', \varepsilon_t'\right)'$ and B conforms with (6). In equation (7), $\underline{\varepsilon}_{t+1}$ contains an excess shock that is a priori recoverable from past observations (Chahrour and Jurado, 2022).[2]

The analysis above shows that we can reinterpret the timing of the JJ framework through a standard SVEC (7): the authority on policy does not exert a control at time t that modifies X_t into X_t^{ctr}. Instead, it observes X_t and introduces a direct shock to the system at time $t+1$ that relates to the observables at t. The intervention adds a shock W_{t+1} to the system that, as it does not correlate with ε_{t+1}, does not render the variance of the innovations singular.

Within the framework of random assignment, it is reasonable to consider the observed process as X_t^{new}. Under this interpretation, we avoid treating the controlled process as the sole observable — indeed, it is not explicitly defined here. However, a key question remains: how does the authority introduce this new shock to the system, a shock that shifts X_{t+1}^{new} without fully controlling it (in contrast to X_t^{ctr} in JJ)? Intuitively, the natural route to achieving such a result relies on considering that the authority uses a primary tool that differs from the observable — partially controlled — policy instrument. This setting was explicitly considered by JJ, and we can further explore it from the econometrician's perspective as a natural approach to addressing the issue of observables within the controlled VAR system.

A key aspect of our discussion centers on equation (7), which demonstrates that the original (α, β) parameters can empirically be recovered through the VARMA structure (see Funovits, 2024). Consequently, the econometrician can perform *ex post* (*i.e.*, after policy implementation) the analysis that the policy maker conducts *ex ante*. The algorithm for maximum likelihood estimation of the parameters of (7) remains a subject for future research; instead, the empirical study in Section 4 illustrates *ex post* policy evaluation in the context of intermediate and final policy targets, which are explored in the next section.

3 Policy with intermediate and final targets

In this section we further explore the controlled cointegrated model to clarify how the policy-making authority can implement their policies. We focus on situations where they cannot directly control the target through their instrument but must instead rely on market forces via an *intermediate* target. This issue was previously considered in JJ, but here we demonstrate how an applied econometrician can assess and identify these targets. The selection of final and intermediate policy target variables presents a challenge both for policymakers and econometricians analysing time series data. To ensure tractability, we restrict ourselves to the case where $m = 1$, meaning there is a singular policy target along with a singular policy

[2]We do not assess here whether the shocks are actually recoverable in all situations.

instrument in the CVAR system. If the *final* target is recognised as $b'X_t$, the *intermediate* target is then stated as follows, according to JJ:

Definition 3.1 *The intermediate target $c'X_t$ for $c \in \mathbf{R}^p$ is a variable which is cointegrated with the final target $b'X_t$, so that there exists a stationary relationship $b'X_t + \phi c'X_t$ for $\phi \neq 0$.*

Applied economists usually find it possible to determine known vectors b and c along with a, guided by some prior knowledge on conceivable transmission mechanisms of economic policy. It is indeed straightforward to fix a, or to choose an instrument variable, on the basis of a policy tool available to monetary and fiscal authorities. However, selecting b and c in the context of an empirical study is considered more challenging, as either $b'X_t$ or $c'X_t$ may serve as final or intermediate targets. Since insights from economic theory are often insufficient to fully justify the selection process, it is important to devise some data-driven approach. The following subsections will discuss this approach in detail.

3.1 Identifying targets

Let us first consider a procedure for the empirical identification of the two types of policy targets. For this, we see that Definition 3.1 requires there exists $j \leq r$ such that

$$\text{sp}(\beta_j) = \text{sp}(b + \phi c) \quad \text{for} \quad \phi \neq 0,$$

where $\text{sp}(\cdot)$ denotes the vector space spanned by $\cdot$ and β_j is one of the cointegrating vectors in $\beta = (\beta_1, ..., \beta_r)$. We then introduce below our definition of a *final* target on the basis of the selection vector b.

Definition 3.2 *Suppose that $\text{sp}(\beta_j) = \text{sp}(b + \phi c)$ holds for $\phi \neq 0$, along with $b = e_j$, where e_j denotes the j-th column vector of I_p for $j \leq r$ and with $c = e_l$ for $j \neq l$ for $l \leq p$. Let $\alpha = (\alpha_1, ..., \alpha_r)$ be expressed in accordance with β. The variable $b'X_t$ is defined as the final target if $\text{sp}(\alpha_j) = \text{sp}(b)$.*

The definition of the intermediate target, Definition 3.1, then follows from Definition 3.2. Note that b and c are orthogonal unit vectors. In order to understand Definition 3.2, it is important to recognise that $Cb = 0$ is considered a critical attribute of the final target. As an example, if $p = 4$, $r = 2$, $j = 1$ and $l = 2$, it then follows that $b = (1, 0, 0, 0)$ and $c = (0, 1, 0, 0)$, and $Cb = 0$ is equivalent to

$$C = \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

In other words, $b'X_t$ can only serve as a target variable, not as a policy instrument. Note that $Cc = 0$ can also be satisfied; that is, it is compatible with $Cb = 0$. This compatibility requires an additional condition, which is embodied in the condition on α, as shown in Definition 3.2. If we split $\alpha = (\alpha_1, \alpha_2)$ for $r = 2$, the condition $\mathrm{sp}(\alpha_j) = \mathrm{sp}(b)$ implies

$$\alpha = (\alpha_1, \alpha_2) = \begin{pmatrix} \xi & * \\ 0 & * \\ 0 & * \\ 0 & * \end{pmatrix}$$

for $\xi \neq 0$. In other words, $b'X_t$ reacts solely to the disequilibria of $(b + \phi c)' X_{t-1}$. It also ensures $Cb = \beta_\perp(\alpha_\perp' \Gamma \beta_\perp)^{-1} \alpha_\perp' b = 0$. Consequently, even if $Cc = 0$ is also true, only $\mathrm{sp}(\alpha_1) \neq \mathrm{sp}(c)$ is compatible with $\mathrm{sp}(\alpha_1) = \mathrm{sp}(b)$, so that the classification is determined.

We summarise the above argument in a proposition below, assuming that the policy only considers controls as unique variables, not linear combinations thereof.

Proposition 3.3 *Suppose $a = e_i$, $b = e_j$ and $c = e_k$ for $i \neq j \neq k$, $j \leq r$ and $i, k \leq p$. If $a'X_t$ is treated as the policy instrument, $b'X_t$ is identified as the final policy target while $c'X_t$ as the intermediate policy target if the following conditions are satisfied:*

1. $\mathrm{sp}(\beta_j) = \mathrm{sp}(b + \phi c) \quad for \quad \phi \neq 0,$
2. $\mathrm{sp}(\alpha_j) = \mathrm{sp}(b)\,.$

All the conditions here are empirically testable, so that we can treat this proposition as a pre-procedure for CVAR-based policy simulation exercises involving both intermediate and final policy targets. Section 4 below provides an empirical illustration of the procedure.

As a corollary to this proposition, we present the following result:

Corollary 3.4 *Under the conditions of Proposition 3.3, the two vectors $b'C$ and $c'C$ are collinear, along with $b'Cb = c'Cb = 0$.*

Proof. See the Appendix. ■

This corollary has two interesting implications. First, it implies that $c'Ca \neq 0$ means $b'Ca \neq 0$ and *vice versa*, as indicated by JJ. Second, it can facilitate a SVEC-type analysis. In order to explain this second aspect, let us provide the Granger-Johansen representation in the context of the simplified model (2):

$$X_t = C \sum_{j=1}^{t} \varepsilon_i + \sum_{j=1}^{\infty} C_i^* \varepsilon_{t-i} + CX_0 - \alpha (\beta' \alpha) \mu,$$

where $\sum_{j=1}^{\infty} C_i^* \varepsilon_{t-i}$ represents a linear process with the matrices C_i^* decreasing exponentially fast. For the purpose of considering its structural interpretation in the context of SVEC formulation, we introduce a non-singular matrix G so that we can define $u_t = G\varepsilon_t$ for $G\Omega G' = I_p$ and find

$$X_t = \tilde{C}\sum_{j=1}^{t} u_i + \sum_{j=1}^{\infty} \tilde{C}_i^* u_{t-i} + CX_0 - \alpha\left(\beta'\alpha\right)\mu,$$

for $\tilde{C} = CG^{-1}$ and $\tilde{C}_i^* = C_i^* G^{-1}$. See Juselius (2006, Ch.15) *inter alia*, for further details of this type of formulation. The parameter $\tilde{C}$ represents the long-run impact matrix in this context and needs to be restricted further to claim its structural interpretation. If we continue to use the example $p = 4$, $r = 2$ and $j = 1$, the corollary implies, as a result of matrix rotation,

$$\tilde{C} = \begin{pmatrix} 0 & -\phi c_{22} & -\phi c_{23} & -\phi c_{24} \\ 0 & c_{22} & c_{23} & c_{24} \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} G^{-1}. \tag{8}$$

The presence of collinear rows in (8) suggests that the SVEC model can be interpreted such that it is the intermediate target that is directly influenced by a series of long-run structural shocks (permanent shocks), while the final target reflects the shocks via the intermediate target. The collinear structure can reduce the number of parameters in $\tilde{C}$, thereby facilitating its identification within the SVEC framework.

3.2 New process in the classification of policy targets

The arguments presented in the preceding subsection suggest that the derived *new* system can be reformulated to reveal the underlying structure shaped by the implementation of economic policy. The expression of the *new* system is provided in the next proposition, which is based on the simplified CVAR model (2) for the sake of simplicity.

Proposition 3.5 *Suppose that all the conditions in Proposition 3.3 are satisfied, so that $b'X_t$ and $c'X_t$ are identified as the final policy target and the intermediate policy target, respectively. The system for X_t^{new} is then expressed as*

$$\Delta X_{t+1}^{new} = \alpha^\circ\left[(b, c, \delta)' X_t^{new} - \mu^\circ\right] + \varepsilon_{t+1}, \quad for \quad t = k+1, ..., T, \tag{9}$$

where $(b, c, \delta) \in \mathbf{R}^{p\times(r+1)}$ represents a set of cointegrating vectors derived from the rotation of (κ, β) such that

$$(b, c, \delta)' X_t^{new} - \mu^\circ \sim \mathrm{I}(0)$$

and

$$\mathrm{sp}(\delta) = \mathrm{sp}(g_\perp)$$

for $g = (b, c)$, along with a set of adjustment vectors α° and constants μ° derived from $(a + \alpha\beta' a, \alpha)$ and $(\kappa^{*\prime}, \mu)'$ respectively, as a result of the rotation of (κ, β).

Proof. See the Appendix. ■

The derived system (9) explicitly shows that the selection vectors b and c are members of the cointegrating vectors for X_t^{new}, while the remaining cointegrating vectors consist of δ, which is orthogonal to b and c as a consequence of matrix rotation given b and c. The vectors $\delta' X_t^{new}$ are likely to contain the policy instrument $a' X_t^{new}$ as its constituent. Note that the cointegrating space for the original process,

$$\beta = [(b + \phi c)\, \omega, \eta] \in \mathbf{R}^{p \times r}$$

for a scalar ω and $\eta = (\beta_{2,} ..., \beta_r)$, is expanded to

$$\beta^{\circ} \equiv (b, c, \delta) \in \mathbf{R}^{p \times (r+1)}$$

for the *new* process. This expansion suggests that greater stability has been achieved in β° as a result of policy implementation. This aspect is illustrated in the empirical analysis presented in the next section.

4 Empirical application

In this section we provide an empirical application of the above propositions to a series of macroeconomic data of New Zealand. We begin by examining the cointegrating rank of an empirical VAR system and then applies the suggested procedure to the data to distinguish between intermediate and final policy targets in the context of inflation targeting. We also conduct policy simulation exercises using the empirical CVAR system and evaluate the econometrician's ex post assessment thereof.

4.1 Cointegrated VAR

We start with the estimation of an unrestricted VAR model for X_t consisting of New Zealand's quarterly macroeconomic series:

$$X_t = (\pi_t, \pi_t^e, y_t, i_t)',$$

where π_t is a realised annual (year-on-year) inflation rate, π_t^e is a survey-based annual inflation expectation, y_t is the log of real output, i_t is the short-term interest rate. Further details of the data are provided in the Appendix. Although the inclusion of the Monetary Conditions Index (MCI) in X_t was considered, it has been excluded from our analysis. This decision reflects the fact that its publication on the RBNZ website was discontinued at the end of November 2000, signaling its diminished role in policy decisions. The estimation period covers the third quarter of 1992 to the first quarter of 2020, comprising a total of 111 observations. The endpoint coincides with the onset of the COVID-19 pandemic, which significantly impacted New Zealand's economy, so that the affected period has been excluded from the estimation sample.

Figure 5 presents an overview of the data for the four variables. All the series appear to be non-stationary; notably, π_t and π_t^e have exhibited synchronised movements, accompanied by a clear upward trend in y_t. We thus deem it suitable to employ a trend-restricted I(1) CVAR method for the analysis of the data.

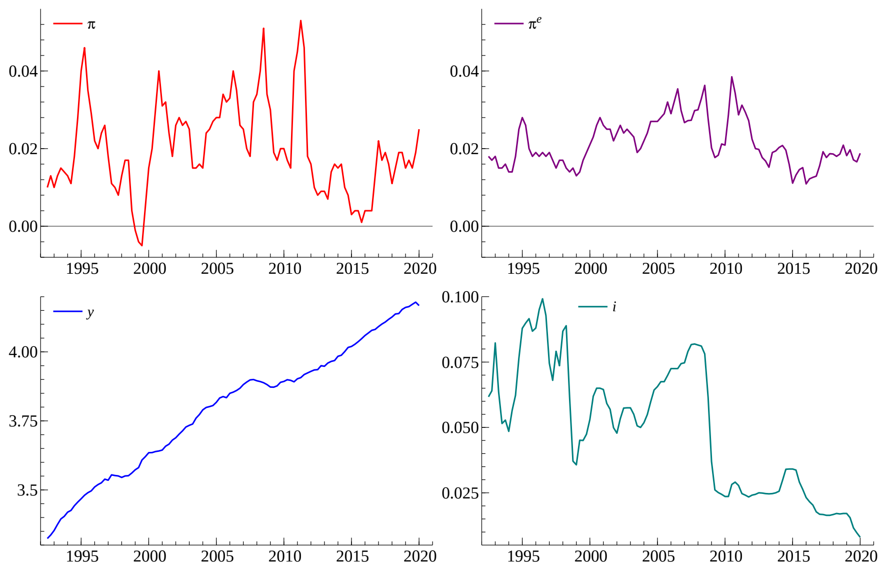

Figure 5: Overview of the data

According to a preliminary regression analysis some of the lagged dynamic terms at $k = 4$ are judged to be fairly significant, resulting in the selection of a VAR(4) model for further study. Figure 6 displays a battery of diagnostic graphs calculated from the estimated VAR(4) model: scaled residuals (the first column), residual autocorrelation functions (ACF, the second column) and residual quantile-quantile plots against normality (QQ plot, the third column). The residuals appear to be free from serial correlations, providing evidence in support of a quasi likelihood-based analysis of cointegration studied by Kurita and Nielsen (2019). The cointegration literature also shows that trace tests for the selection of cointegrating rank are robust to non-normality in the innovation term; see Cheung and Lai (1993), *inter alia*, for further details. The evidence recorded in the figure thus justifies using the VAR(4) model as a basis for exploring the underlying cointegrating rank.

Table 1 reports a class of trace test statistics for the choice of r, $\log LR(r|p)$ for $r = 0, ..., 3$ given $p = 4$. The series of tests are in support of $r = 2$ at the 5% level, so we select this value as the retained cointegration rank. We then proceed to applying the procedure based on Proposition 3.3, for which we recall that likelihood ratio tests for restrictions on α and β

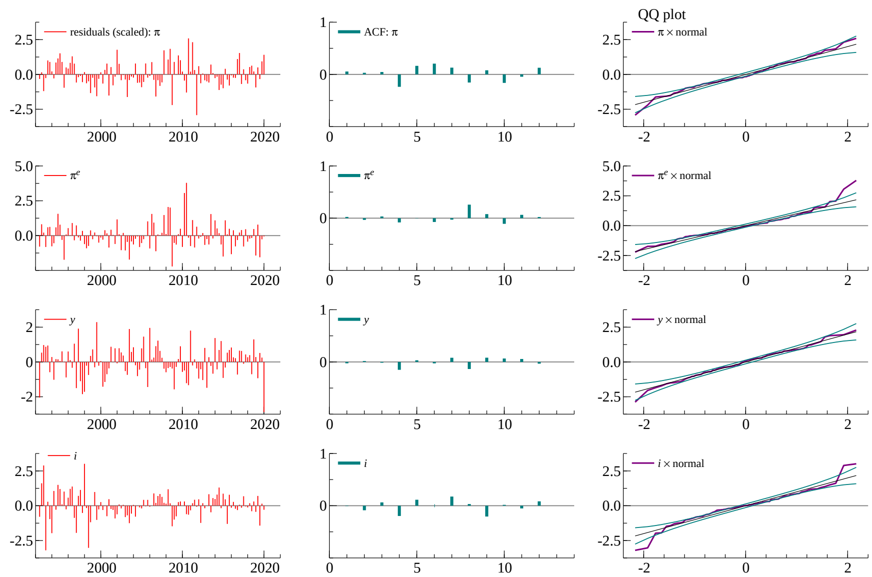

Figure 6: Residual diagnostics

Table 1: Inference on the cointegrating rank for X_t.

	$r = 0$	$r \leq 1$	$r \leq 2$	$r \leq 3$
log LR	84.738[0.000]**	45.168[0.027]*	21.292[0.169]	6.815[0.375]

Note. Figures in square brackets are p-values.

** and * denote significance at the 1% and the 5% level, respectively.

have asymptotic χ^2 distributions, given the selection of cointegrating rank (see Johansen, 1996, Chs. 7 and 8 for further details).

4.2 Classifying the policy targets

The hypothetical long-run structure we envision is as follows: (i) expected inflation serves as the intermediate target while actual inflation is the final target, resulting in the long-run synchronisation of the two inflation rates, and (ii) expected inflation is driven by the output gap and interest rate, leading to a long-run Phillips curve formulation. Given this hypothetical structure as well as the selection of the interest rate as an instrument variable, we conceive the specification of $a = (0, 0, 0, 1)'$, $b = (1, 0, 0, 0)'$, $c = (0, 1, 0, 0)'$ and $\phi = -1$; that is, $a'X_t = i_t$, $b'X_t = \pi_t$, $c'X_t = \pi_t^e$ and $(b - c)X_t = \pi_t - \pi_t^e$. The parameter ϕ can be estimated in the CVAR framework but it seems natural to preset $\phi = -1$ as a hypothesis, suggesting a presumed synchronisation of π_t and π_t^e. This specification then allows us to test for the validity of the two hypotheses given in Proposition 3.3, according to which we should fail to reject both of them. The first hypothesis to be tested under the above specification is

$$\mathrm{H}_0^{(1)} : \mathrm{sp}\,(\beta_1) = \mathrm{sp}\,(b - c)\,.$$

In order to identify the cointegrating space, we have also introduced a normalisation scheme for the second cointegrating vector (0 and 1 for the first and the second element, respectively), arriving at the following estimates:

$$\hat{\alpha}\begin{pmatrix}\hat{\beta}\\ \hat{\rho}\end{pmatrix}'\begin{pmatrix}X_{t-1}\\ t\end{pmatrix} = \begin{bmatrix} \underset{(0.097)}{-0.449} & \underset{(0.101)}{-0.152} \\ \underset{(0.043)}{0.013} & \underset{(0.045)}{-0.218} \\ \underset{(0.120)}{-0.062} & \underset{(0.125)}{-0.179} \\ \underset{(0.095)}{0.003} & \underset{(0.099)}{-0.147} \end{bmatrix} \begin{bmatrix} \underset{(-)}{1} & \underset{(-)}{0} \\ \underset{(-)}{-1} & \underset{(-)}{1} \\ \underset{(-)}{0} & \underset{(0.045)}{-0.240} \\ \underset{(-)}{0} & \underset{(0.116)}{0.494} \\ \underset{(-)}{0} & \underset{(0.0004)}{0.002} \end{bmatrix}' \begin{pmatrix} \pi_{t-1} \\ \pi^e_{t-1} \\ y_{t-1} \\ i_{t-1} \\ t \end{pmatrix}.$$

The log-likelihood ratio test statistic ($\log LR$) is 4.046[0.257] with its p-value, according to $\chi^2(3)$, given in the square brackets, so that $\mathrm{H}_0^{(1)}$ is not rejected at conventional levels.

Next is the testing of

$$\mathrm{H}_0^{(2)} : \mathrm{sp}\,(\alpha_1) = \mathrm{sp}\,(b)\,,$$

under $\mathrm{H}_0^{(1)}$. Imposing a set of additional restrictions consistent with $\mathrm{H}_0^{(2)}$ yields

$$\hat{\alpha}\begin{pmatrix}\hat{\beta}\\ \hat{\rho}\end{pmatrix}'\begin{pmatrix}X_{t-1}\\ t\end{pmatrix} = \begin{bmatrix} \underset{(0.092)}{-0.462} & \underset{(0.099)}{-0.146} \\ \underset{(-)}{0} & \underset{(0.043)}{-0.210} \\ \underset{(-)}{0} & \underset{(0.118)}{-0.202} \\ \underset{(-)}{0} & \underset{(0.094)}{-0.148} \end{bmatrix} \begin{bmatrix} \underset{(-)}{1} & \underset{(-)}{0} \\ \underset{(-)}{-1} & \underset{(-)}{1} \\ \underset{(-)}{0} & \underset{(0.045)}{-0.241} \\ \underset{(-)}{0} & \underset{(0.118)}{0.508} \\ \underset{(-)}{0} & \underset{(0.0003)}{0.002} \end{bmatrix}' \begin{pmatrix} \pi_{t-1} \\ \pi^e_{t-1} \\ y_{t-1} \\ i_{t-1} \\ t \end{pmatrix}, \tag{10}$$

along with $\log LR = 4.395[0.623]$ on the basis of $\chi^2\,(6)$, hence leading to the conclusion that $\mathrm{H}_0^{(2)}$ fails to be rejected.

In order to consolidate the findings so far, we will also check the *rejection* of

$$\mathrm{H}_0^{(3)} : \mathrm{sp}\,(\alpha_1) = \mathrm{sp}\,(c)$$

under $\mathrm{H}_0^{(1)}$, so that $\mathrm{sp}\,(\alpha_1) \neq \mathrm{sp}\,(c)$ is ensured. The resulting estimates are given below:

$$\hat{\alpha}\begin{pmatrix}\hat{\beta}\\ \hat{\rho}\end{pmatrix}'\begin{pmatrix}X_{t-1}\\ t\end{pmatrix} = \begin{bmatrix} \underset{(-)}{0} & \underset{(0.115)}{-0.312} \\ \underset{(0.041)}{0.081} & \underset{(0.049)}{-0.254} \\ \underset{(-)}{0} & \underset{(0.128)}{-0.233} \\ \underset{(-)}{0} & \underset{(0.102)}{-0.137} \end{bmatrix} \begin{bmatrix} \underset{(-)}{1} & \underset{(-)}{0} \\ \underset{(-)}{-1} & \underset{(-)}{1} \\ \underset{(-)}{0} & \underset{(0.038)}{-0.199} \\ \underset{(-)}{0} & \underset{(0.098)}{0.433} \\ \underset{(-)}{0} & \underset{(0.0003)}{0.002} \end{bmatrix}' \begin{pmatrix} \pi_{t-1} \\ \pi^e_{t-1} \\ y_{t-1} \\ i^s_{t-1} \\ t \end{pmatrix}.$$

The corresponding test statistic is $\log LR = 26.846[0.0002]^{**}$ according to $\chi^2(6)$, strong evidence against $\mathrm{H}_0^{(3)}$, so we are able to reject this hypothesis. Overall, we conclude that $b'X_t = \pi_t$ is identified as the final policy target while $c'X_t = \pi_t^e$ as the intermediate policy target.

Finally, getting back to (10), we introduce a zero restriction on the first element of the second adjustment vector, so the feedback mechanism is consistent with the identification scheme for the cointegrating space:

$$\hat{\alpha}\begin{pmatrix}\hat{\beta}\\ \hat{\rho}\end{pmatrix}'\begin{pmatrix}X_{t-1}\\ t\end{pmatrix} = \begin{bmatrix} \underset{(0.089)}{-0.502} & \underset{(-)}{0} \\ \underset{(-)}{0} & \underset{(0.0411)}{-0.193} \\ \underset{(-)}{0} & \underset{(0.118)}{-0.238} \\ \underset{(-)}{0} & \underset{(0.094)}{-0.139} \end{bmatrix} \begin{bmatrix} \underset{(-)}{1} & \underset{(-)}{0} \\ \underset{(-)}{-1} & \underset{(-)}{1} \\ \underset{(-)}{0} & \underset{(0.046)}{-0.224} \\ \underset{(-)}{0} & \underset{(0.119)}{0.497} \\ \underset{(-)}{0} & \underset{(0.0004)}{0.002} \end{bmatrix}' \begin{pmatrix} \pi_{t-1} \\ \pi_{t-1}^e \\ y_{t-1} \\ i_{t-1} \\ t \end{pmatrix}, \tag{11}$$

along with $\log LR = 6.575[0.474]$, hence $\mathrm{H}_0^{(2)}$ being non-rejected according to $\chi^2(7)$. The identified structure in (11) indicate clearly how the instrument affects the intermediate target, thus having an influence on the final policy target.

The condition for the controllability of $c'X_t = \pi_t^e$ by means of $a'X_t = i_t$ is given as $c'Ca \neq 0$, which implies $b'Ca = c'Ca \neq 0$; see Corollary 3.4. The parameter estimates $\hat{\alpha}$ and $(\hat{\beta}', \hat{\rho}')$ recorded in (11) have been used in the estimation of the C matrix:

$$\hat{C} = \begin{bmatrix} \underset{(-)}{0} & \underset{(-)}{0.206} & \underset{(-)}{0.038} & \underset{(-)}{-0.351} \\ \underset{(-)}{0} & \underset{(0.134)}{0.206} & \underset{(0.057)}{0.038} & \underset{(\mathbf{0.072})}{\mathbf{-0.351}} \\ \underset{(-)}{0} & \underset{(0.868)}{-1.695} & \underset{(0.370)}{1.525} & \underset{(0.467)}{-0.260} \\ \underset{(-)}{0} & \underset{(0.445)}{-1.179} & \underset{(0.190)}{0.612} & \underset{(0.239)}{0.589} \end{bmatrix},$$

in which figures in parentheses denote standard errors. Inference concerning $\hat{C}$ is made on the basis of Paruolo (1997). The element $\hat{C}_{24}$ in bold corresponds to $c'\hat{C}a$, which is judged to be significantly different from 0 at the conventional level. Moreover, its value is negative ($c'\hat{C}a < 0$), indicating that an increase in the short-term interest rate results in a decrease in expected inflation. The first and second rows of $\hat{C}$ (that is, $b'C$ and $c'C$) are identical along with the first column zero, aligned with Corollary 3.4, covering the identity $c'\hat{C}a = \hat{C}_{24} = \hat{C}_{14} = b'\hat{C}a$.

A series of zeros in the first column of $\hat{C}$ indicates π_t lacks the capability to influence all the other variables in the system, thus categorising π_t as the final target in the context of policy control, while $c'\hat{C}a \neq 0$ indicates π_t^e can be controlled by i_t. We are thus justified in concluding that the instrument $a'X_t = i_t$ is employed to control the intermediate target

$c'X_t = \pi_t^e$, which is cointegrated with the final target $b'X_t = \pi_t$, so that the final target is also controllable by the instrument by way of the intermediate target.

4.3 Empirical study of X_t^{new} for *ex post* theory verification

We now employ the empirical CVAR model obtained above to simulate a class of *new* processes X_t^{new} subject to the control rule. The argument in Section 2 allows us to focus on a simulation study of X_t^{new}, instead of the two processes (X_t^{new}, X_t^{ctr}). Following the argument of Section 3, we have demonstrated above the controllability of $c'X_t = \pi_t^e$ by means of $a'X_t = i_t$, with the consequence that $b'X_t = \pi_t$ is also under control. We are thus justified in conducting a simulation exercise using X_t^{new}, with the aim of controlling inflation expectations rather than actual inflation. We will perform a set of two simulation studies here so as to substantiate the theoretical arguments outlined earlier.

First, we simulate X_t^{new} using an empirical version of the extended system (3) under the natural control rule, where the selection vector c is used in place of b, with a pre-set target value of $c^* = 0.015$. This target value is intentionally set below 0.02 (*i.e.*, 2%), the actual target rate adopted by the RBNZ, to demonstrate the workings of the policy simulation and its informativeness. Figure 7(a) displays the *new* instrument $a'X_t^{new} = i_t^{new}$ under the projected policy, alongside the actual $a'X_t = i_t$; The former tends to exceed the latter, reflecting the responses of tighter monetary policy required to achieve the target value $c^* = 0.015$ in a counterfactual scenario. This monetary contraction has caused $c'X_t^{new} = \pi_t^{e,new}$ to hover around the target level, as shown in Figure 7(b); the *new* series $\pi_t^{e,new}$ appears to be stationary with a mean of $c^* = 0.015$, in contrast to the actual $c'X_t = \pi_t^e$, which exhibits more a clearly non-stationary behaviour than $\pi_t^{e,new}$. We can therefore conclude that expected inflation can be manipulated to achieve its pre-specified target level in the counterfactual world through the use of the short-term rate instrument, consistent with the expectation derived from $c'\hat{C}a < 0$ discussed in the previous subsection. This also implies that actual inflation can be controlled due to its synchronisation with the expected inflation rate.

Table 2: Inference on the cointegrating rank for X_t^{new}.

	$r = 0$	$r \leq 1$	$r \leq 2$	$r \leq 3$
log LR	128.05[0.000]**	63.861[0.000]**	33.007[0.000]**	9.118[0.178]

Note. Figures in square brackets are p-values.

** denotes significance at the 1% level.

Second, we place ourselves in the position of an applied econometrician within this counterfactual policy environment and pose the following question: Would their empirical analysis be able to detect the policy? To answer this, we conduct a cointegration study of

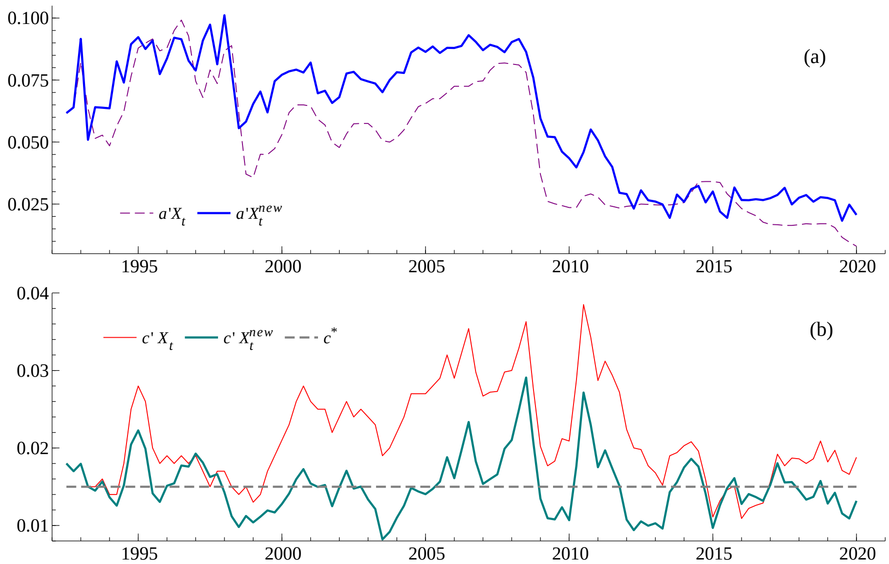

Figure 7: Policy simulation

X_t^{new} derived from the above simulation to verify Proposition 3.5. Table 2 presents, in the same manner as Table 1, a set of trace test statistics calculated from the generated X_t^{new} series under the the pre-specified target level $c^* = 0.015$. The results provide strong evidence supporting $r = 3$, as predicted, contrasting with $r = 2$ recorded in Table 1.

The selection of $r = 3$ enables us to further explore whether a cointegrating structure consistent with Proposition 3.5 truly underlies the system for X_t^{new}. The revealed cointegrating relationships (apart from the linear trend) are

$$(b, c, \delta)' \, X_t^{new} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \underset{(0.059)}{-2.223} \end{bmatrix}' \begin{pmatrix} \pi_{t-1}^{new} \\ \pi_{t-1}^{e,new} \\ y_{t-1}^{new} \\ i_{t-1}^{new} \end{pmatrix},$$

and the test statistic is $\log LR = 4.20[0.380]$ according to $\chi^2(4)$, thus accepting the null of the joint restrictions. The revealed structure aligns with the predictions in Proposition 3.5, representing a counterfactual world where both actual and expected inflation rates have become stationary as a consequence of a series of policy interventions. This is accompanied by the stationary combination of y_{t-1}^{new} and i_{t-1}^{new} alone (not including $\pi_{t-1}^{e,new}$), which contrasts with the second cointegrating relationship in (11) that consists of π_{t-1}^{e}, y_{t-1} and i_{t-1}.

5 Conclusion

This paper explores the consequences of CVAR-based control theory from an econometrician's perspective, focusing on model estimation and counterfactual analysis. By reexamining the mechanisms underlying JJ's theoretical results, we discuss the challenges faced by an applied econometrician. Monte Carlo studies illustrate the statistical properties of *new* and *controlled* processes, reinforcing the argument that inference based on the controlled process is unreliable. We also show that the same mathematical results can be interpreted under a different policy timing narrative. In this context, the timing of the JJ framework can be reinterpreted within an SVEC model. Rather than modifying policy variables contemporaneously, the policy-making authority introduces a structural shock that is fully recoverable from past observations, thereby avoiding singularity in the variance of the innovations to the observables. Under this random assignment interpretation, the observed process should correspond to the *new* process. Additionally, this paper presents a data-driven procedure for categorizing intermediate and final policy targets within a model framework. The effectiveness of this procedure is demonstrated through an analysis of New Zealand's monetary policy data. While the algorithm for maximum likelihood estimation of SVEC parameters remains a topic for future research, the empirical study illustrates *ex post* policy evaluation in the context of intermediate and final policy targets. This paper aims to lay the foundation for future research that enhances the practicality of CVAR-based control analysis in applied macroeconomic and financial studies.

Acknowledgements:

We are sincerely grateful to S. Johansen and K. Juselius for their communication with us on CVAR-based control theory during the Workshop to Celebrate 40 Years of Unit Roots and Cointegration held in Oxford. We would like to thank N. Ericsson for his helpful suggestion and other workshop participants for their interest in our study. We also wish to express our gratitude to A. Banerjee, S. Mavroeidis, R. Taylor, and E. Heritage for organising this workshop. Kurita would like to gratefully acknowledge financial support from JSPS KAKENHI (Grant Number: 24K04877).

Appendix

A Proofs referring to Sections 2 and 3:

Derivation of the controlled process in Section 2.2.2:

The control rule is given by

$$\kappa' = -\left(b'C\overline{a}\right)^{-1} b'C \quad \text{and} \quad \kappa^* = -\left(b'C\overline{a}\right)^{-1} \left[b^* - b'\alpha\left(\beta'\alpha\right)^{-1}\mu\right],$$

so by construction

$$\kappa'\alpha = -\left(b'C\overline{a}\right)^{-1} b'C\alpha = 0 \quad \text{and} \quad \kappa'\overline{a} = -\left(b'C\overline{a}\right)^{-1} b'C\overline{a} = -I_m.$$

Now, consider $x_{t+1}^{ctr} = \left(I_p + \overline{a}\kappa'\right) x_{t+1}^{new} - \overline{a}\kappa^*$ with $\kappa'\alpha = 0$ and $I_m + \kappa'\overline{a} = 0$. Using equation (3) above, which is the dynamics of the *new* process,

$$\Delta x_{t+1}^{new} = \left[\alpha, \left(I_p + \alpha\beta'\right)\overline{a}\right] \begin{pmatrix} \beta' x_t^{new} - \mu \\ \kappa' x_t^{new} - \kappa^* \end{pmatrix} + \varepsilon_{t+1},$$

we see that

$$\Delta x_{t+1}^{ctr} = \left(I_p + \overline{a}\kappa'\right)\Delta x_{t+1}^{new} = \left(I_p + \overline{a}\kappa'\right)\left[\alpha, \left(I_p + \alpha\beta'\right)\overline{a}\right] \begin{pmatrix} \beta' x_t^{new} - \mu \\ \kappa' x_t^{new} - \kappa^* \end{pmatrix} + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1},$$

where $\left(I_p + \overline{a}\kappa'\right)\alpha = \alpha$, and

$$\begin{aligned} \left(I_p + \overline{a}\kappa'\right)\left(I_p + \alpha\beta'\right)\overline{a} &= \left(I_p + \alpha\beta' + \overline{a}\kappa'\right)\overline{a}, \\ &= \left(I_p + \alpha\beta'\right)\overline{a} - \overline{a}. \end{aligned}$$

Hence,

$$\left(I_p + \overline{a}\kappa'\right)\left(\alpha, \left(I_p + \alpha\beta'\right)\overline{a}\right) = \alpha\left(I_r, \beta'\overline{a}\right).$$

The *controlled* process therefore becomes

$$\begin{aligned} \Delta x_{t+1}^{ctr} &= \alpha\left(I_r, \beta'\overline{a}\right) \begin{pmatrix} \beta' x_t^{new} - \mu \\ \kappa' x_t^{new} - \kappa^* \end{pmatrix} + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1} \\ &= \alpha\left[\beta' x_t^{new} - \mu + \beta'\overline{a}\left(\kappa' x_t^{new} - \kappa^*\right)\right] + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1} \\ &= \alpha\left[\beta'\left(I_p + \overline{a}\kappa'\right)x_t^{new} - \mu\right] + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1} \\ \Delta x_{t+1}^{ctr} &= \alpha\left(\beta' x_t^{ctr} - \mu\right) + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1}, \end{aligned}$$

i.e., although the *controlled* process seems to exhibit one extra cointegration relation, the latter is by construction proportional to the original one so it is degenerate. In other words, the controlled process can be expressed as

$$\Delta x_{t+1}^{ctr} = \alpha\left(I_r, \overline{a}\right) \begin{pmatrix} \beta' x_t^{ctr} - \mu \\ \kappa' x_t^{ctr} - \kappa^* \end{pmatrix} + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1},$$

and pre-multiplying this expression by κ' and performing further manipulation yields the following identity:

$$\begin{aligned}\kappa' x^{ctr}_{t+1} &= \kappa' x^{ctr}_t + \kappa'\overline{a}\left(\kappa' x^{ctr}_t - \kappa^*\right) + \kappa'\left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1} \\ &= \left(I_p + \kappa'\overline{a}\right)\kappa' x^{ctr}_t - \kappa'\overline{a}\kappa^* + \left(I_p + \kappa'\overline{a}\right)\kappa'\varepsilon_{t+1} \\ &= \kappa^*,\end{aligned}$$

due to $\kappa'\left(I_p + \overline{a}\kappa'\right) = \left(I_p + \kappa'\overline{a}\right)\kappa' = 0$. It then follows that $\kappa'\Delta x^{ctr}_{t+1} = 0$. The matrix $\left(I_p + \overline{a}\kappa'\right)\Omega\left(I_p + \overline{a}\kappa'\right)'$ is singular since

$$\left(I_p + \overline{a}\kappa'\right)\Omega\left(I_p + \overline{a}\kappa'\right)'\kappa = 0,$$

i.e., the matrix has a zero eigenvalue and its determinant is zero.

Next, we extend the above to a general CVAR system with $k > 1$. Without loss of generality, we fix $k = 3$ and provide the system in companion form by following JJ:

$$\Delta\tilde{X}_{t+1} = \tilde{\alpha}\left(\tilde{\beta}'\tilde{X}_t - \tilde{\mu}\right) + \tilde{\varepsilon}_{t+1},$$

where

$$\tilde{X}_{t+1} = \begin{pmatrix} X_{t+1} \\ X_t \\ X_{t-1} \end{pmatrix}, \quad \tilde{\varepsilon}_{t+1} = \begin{pmatrix} \varepsilon_{t+1} \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mu} = \begin{pmatrix} \mu \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{\alpha} = \begin{pmatrix} \alpha & \Gamma_1 & \Gamma_2 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{pmatrix} \quad \text{and} \quad \tilde{\beta} = \begin{pmatrix} \beta & I_p & 0 \\ 0 & -I_p & I_p \\ 0 & 0 & -I_p \end{pmatrix},$$

so that we find

$$\tilde{\alpha}_\perp = \left(\alpha'_\perp, -\alpha'_\perp\Gamma_1, -\alpha'_\perp\Gamma_2\right)' \quad \text{and} \quad \tilde{\beta}_\perp = \left(\beta'_\perp, \beta'_\perp, \beta'_\perp\right)'.$$

In addition, recalling the definition $C = \beta_\perp(\alpha'_\perp\Gamma\beta_\perp)^{-1}\alpha'_\perp$ when $k > 1$, we introduce

$$\tilde{a} = (a', 0, 0)' \quad \tilde{b} = (b', 0, 0)' \quad \text{and} \quad \tilde{\kappa} = \left(\kappa'_1, \kappa'_2, \kappa'_3\right)',$$

for $\kappa'_1 = -\left(b'C\overline{a}\right)^{-1} b'C,\ \kappa'_2 = -\kappa'_1\Gamma_1$ and $\kappa'_3 = -\kappa'_1\Gamma_2$ so that $\tilde{\kappa}'\tilde{\alpha} = 0$ holds. Define

$$\tilde{X}^{new}_{t+1} = \left(X^{new\prime}_{t+1}, X^{ctr\prime}_t, X^{ctr\prime}_{t-1}\right)' \quad \text{and} \quad \tilde{X}^{ctr}_{t+1} = \left(X^{ctr\prime}_{t+1}, X^{ctr\prime}_t, X^{ctr\prime}_{t-1}\right)',$$

which are driven by

$$\tilde{X}^{new}_{t+1} = \left(I_{3p} + \tilde{\alpha}\tilde{\beta}'\right)\tilde{X}^{ctr}_t - \tilde{\alpha}\tilde{\mu} + \tilde{\varepsilon}_{t+1}, \qquad (12)$$

$$\tilde{X}^{ctr}_{t+1} = \tilde{X}^{new}_{t+1} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\left(\tilde{\kappa}'\tilde{X}^{new}_{t+1} - \kappa^*\right), \qquad (13)$$

for $\kappa^* = -(b'C\overline{a})^{-1}[b^* - b'(I_p - C\Gamma)\overline{\beta}\mu]$. Substituting (12) into (13) leads to

$$\begin{aligned}\tilde{X}_{t+1}^{ctr} &= \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\left[\left(I_{3p} + \tilde{\alpha}\tilde{\beta}'\right)\tilde{X}_t^{ctr} - \tilde{\alpha}\tilde{\mu} + \tilde{\varepsilon}_{t+1}\right] - \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa^* \\ &= \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\left(I_{3p} + \tilde{\alpha}\tilde{\beta}'\right)\tilde{X}_t^{ctr} - \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\alpha}\tilde{\mu} \\ &\qquad - \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa^* + \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\varepsilon}_{t+1}.\end{aligned}$$

This is reduced to, due to $\tilde{\kappa}'\tilde{\alpha} = 0$,

$$\tilde{X}_{t+1}^{ctr} = \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}' + \tilde{\alpha}\tilde{\beta}'\right]\tilde{X}_t^{ctr} - \tilde{\alpha}\tilde{\mu} - \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa^* + \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\varepsilon}_{t+1}.$$

Hence, we arrive at

$$\begin{aligned}\Delta\tilde{X}_{t+1}^{ctr} &= \left[\tilde{\alpha}\tilde{\beta}' + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{X}_t^{ctr} - \tilde{\alpha}\tilde{\mu} - \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa^* + \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\varepsilon}_{t+1} \\ &= \left[\tilde{\alpha}\tilde{\beta}', \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\right]\begin{pmatrix}\tilde{\beta}'\tilde{X}_t^{ctr} - \tilde{\mu} \\ \tilde{\kappa}'\tilde{X}_t^{ctr} - \kappa^*\end{pmatrix} + \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\varepsilon}_{t+1}.\end{aligned}$$

Noting that $I_m + \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1} = I_m + \kappa_1' a\left(a'a\right)^{-1} = 0$, we pre-multiply the above equation by $\tilde{\kappa}'$ to derive the identity

$$\begin{aligned}\tilde{\kappa}'\tilde{X}_{t+1}^{ctr} &= \tilde{\kappa}'\tilde{X}_t^{ctr} + \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\left(\tilde{\kappa}'\tilde{X}_t^{ctr} - \kappa^*\right) + \tilde{\kappa}'\left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\widetilde{\varepsilon}_{t+1} \\ &= \left[I_m + \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\right]\tilde{\kappa}'\tilde{X}_t^{ctr} - \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa^* + \left[I_m + \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\right]\tilde{\kappa}'\widetilde{\varepsilon}_{t+1} \\ &= \kappa^*.\end{aligned}$$

Hence,

$$\tilde{\kappa}'\tilde{X}_{t+1}^{ctr} = \kappa_1'\left(X_{t+1}^{ctr} - \Gamma_1 X_t^{ctr} - \Gamma_2 X_{t-1}^{ctr}\right) = \kappa^*,$$

or, equivalently, its general expression covering $k > 3$ is

$$\kappa_1'\Gamma\left(L\right)X_{t+1}^{ctr} = \kappa^*,$$

for $\Gamma\left(L\right) = I_p - \Gamma_1 L - \cdots - \Gamma_k L^k$. ■

Proof of Corollary 3.4:

Referring to the definition of the C matrix, we find

$$\left(b' + \phi c'\right)C = \left(b' + \phi c'\right)\beta_\perp(\alpha_\perp'\Gamma\beta_\perp)^{-1}\alpha_\perp' = 0,$$

which yields the collinearity $b'C = -\phi c'C$ for $\phi \neq 0$. The result $b'Cb = c'Cb = 0$ follows directly from $Cb = 0$. ■

Proof of Proposition 3.5:

The *new* process for $t = k+1, ..., T$ is expressed as

$$\Delta X_{t+1}^{new} = (a + \alpha\beta' a, \alpha) \left[\begin{pmatrix} \kappa' \\ \beta' \end{pmatrix} X_{t+1}^{new} - \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix} \right] + \varepsilon_{t+1}.$$

Recall the set of known vectors: $a = e_i$, $b = e_j$ and $c = e_k$ for $i \neq j \neq k$, $j \leq r$ and $i, k \leq p$. We specify here $b = e_1$ and $c = e_2$, setting $j = 1$ and $i = 2$. Noting the identity $\kappa' = -(b'C\overline{a})^{-1}b'C = -(b'C\overline{a})^{-1}b' + (b'C\overline{a})^{-1}b'\alpha\,(\beta'\alpha)^{-1}\beta'$, we can rewrite the above as

$$\Delta X_{t+1}^{new} = (a + \alpha\beta' a, \alpha)\, R_1 \left[\begin{pmatrix} b' \\ \beta' \end{pmatrix} X_{t+1}^{new} - R_1^{-1} \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix} \right] + \varepsilon_{t+1},$$

where R_1 is a rotation matrix defined as

$$R_1 = \begin{pmatrix} -(b'Ca)^{-1} & (b'C\overline{a})^{-1}b'\alpha\,(\beta'\alpha)^{-1} \\ 0 & I_r \end{pmatrix},$$

and the constant term is subject to

$$e_1' R_1^{-1} \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix} = b^*.$$

Using the condition $\mathrm{sp}\,(\beta_1) = \mathrm{sp}\,(b + \phi c)$ for $\phi \neq 0$, we re-express $\beta = [(b + \phi c)\,\omega, \eta]$ for a non-zero scalar ω and $\eta = (\beta_2, ..., \beta_r)$, along with a conformable decomposition of the constant term, so that we obtain

$$\Delta X_{t+1}^{new} = (a + \alpha\beta' a, \alpha)\, R_1 \left[\begin{pmatrix} b' \\ \omega\,(b' + \phi c') \\ \eta' \end{pmatrix} X_{t+1}^{new} - \begin{pmatrix} b^* \\ \mu_1 \\ \mu_2 \end{pmatrix} \right] + \varepsilon_{t+1},$$

where

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = (0, I_r)\, R_1^{-1} \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix}.$$

Furthermore, as a result of matrix rotation,

$$\Delta X_{t+1}^{new} = (a + \alpha\beta' a, \alpha)\, R_1 R_2 \left[\begin{pmatrix} b' \\ c' \\ \eta' \end{pmatrix} X_{t+1}^{new} - R_2^{-1} \begin{pmatrix} b^* \\ \mu_1 \\ \mu_2 \end{pmatrix} \right] + \varepsilon_{t+1},$$

where

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \omega & \omega\phi & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{r-1} & \\ 0 & 0 & & & \end{pmatrix}.$$

Since b and c are the first and second cointegrating vectors for X_{t+1}^{new}, both of which are linearly independent of η, we can orthogonalise η with respect to $g = (b, c)$. As a result of orthogonalisation, we obtain

$$\Delta X_{t+1}^{new} = (a + \alpha\beta' a, \alpha) R_1 R_2 R_3 \left[\begin{pmatrix} b' \\ c' \\ \delta' \end{pmatrix} X_{t+1}^{new} - R_3^{-1} R_2^{-1} \begin{pmatrix} b^* \\ \mu_1 \\ \mu_2 \end{pmatrix} \right] + \varepsilon_{t+1},$$

where

$$R_3 = \begin{pmatrix} I_2 & 0 \\ \eta' g (g'g)^{-1} & I_{r-1} \end{pmatrix},$$

that is, its inverse R_3^{-1} denotes a matrix that orthogonalises η with respect to g, thereby resulting in $\mathrm{sp}(\delta) = \mathrm{sp}(g_\perp)$. It also follows that

$$\alpha^\circ = (a + \alpha\beta' a, \alpha) R_1 R_2 R_3 \quad \text{and} \quad \mu^\circ = R_3^{-1} R_2^{-1} R_1^{-1} \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix}.$$

■

B Data definitions and sources

Details of the definitions of the data analysed in Section 4 and their sources are provided below.

B.1 Data definitions

π_t = the annual (year-on-year) rate of inflation calculated from the Consumer Price Index (CPI), expressed as a decimal.

π_t^e = the annual rate of expected CPI inflation (1 year out) based on surveys of expectations, expressed as a decimal.

y_t = the log of the production-based real Gross Domestic Product, seasonally adjusted.

i_t = the overnight interbank cash rate, quarterly average of monthly data, expressed as a decimal.

B.2 Sources

All the data were obtained from the website of the Reserve Bank of New Zealand (accessed on 14 June 2024). Detailed sources are as follows:

π_t - https://www.rbnz.govt.nz/statistics/series/economic-indicators/prices

π_t^e - https://www.rbnz.govt.nz/statistics/series/economic-indicators/survey-of-expectations

y_t - https://www.rbnz.govt.nz/statistics/series/economic-indicators/gross-domestic-product

i_t - https://www.rbnz.govt.nz/statistics/series/exchange-and-interest-rates /wholesale-interest-rates

C DGPs in the Monte Carlo and its further outputs

The baseline data-generating process for the study in Section 2.3 is commonly formulated as a CVAR(1) process:

$$\Delta X_t = \alpha\left(\beta' X_{t-1} + \mu\right) + \varepsilon_t,$$

where ε_t is a multivariate *i.i.d.* pseudo normal process, $N(0, d^2\Omega)$. Here, Ω is a positive definite symmetric matrix, each diagonal and off-diagonal element assigned a unit value and a quarter, respectively, along with a damping factor $d = 0.01$. The parameters for the above process as well as a set of selection vectors a and b vary according to p and r as follows:

p	r	α	β	μ
3	1	$(-0.2, 0.1, 0)'$	$(1, -1, 1)'$	-0.01
3	2	$\begin{pmatrix} -0.2 & 0.1 & 0 \\ 0 & -0.1 & 0 \end{pmatrix}'$	$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}'$	$(-0.01, -0.13)'$
4	1	$(-0.2, 0.1, 0, 0)'$	$(1, -1, -0.5, 1)'$	0.015
4	2	$\begin{pmatrix} -0.2 & 0.1 & 0 & 0 \\ 0 & -0.1 & -0.2 & 0 \end{pmatrix}'$	$\begin{pmatrix} 1 & -1 & -0.5 & 1 \\ 0 & 1 & 1 & -0.5 \end{pmatrix}'$	$(0.015, -0.08)'$
4	3	$\begin{pmatrix} -0.2 & 0.1 & 0 & 0 \\ 0 & -0.1 & -0.2 & 0 \\ 0 & 0 & -0.1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & -0.5 & 1 \\ 0 & 1 & 1 & -0.5 \\ 0 & 0 & 1 & -2.0 \end{pmatrix}'$	$(0.015, -0.08, 0.03)'$

p	r	a	b
3	1, 2	$(0, 0, 1)'$	$(1, 0, 0)'$
4	1, 2, 3	$(0, 0, 0, 1)'$	$(1, 0, 0, 0)'$

The parameters above are selected on the basis of a typical empirical study involving inflation rates and short-term interest rates, along with other macroeconomic series. The initial values X_0 range from 0.02 to 0.05, mimicking plausible inflation and interest rates. In each replication of the Monte Carlo study, 30 initial observations are discarded to mitigate the impact of the initial values.

Further figures obtained from the Monte Carlo study are presented below in this appendix.

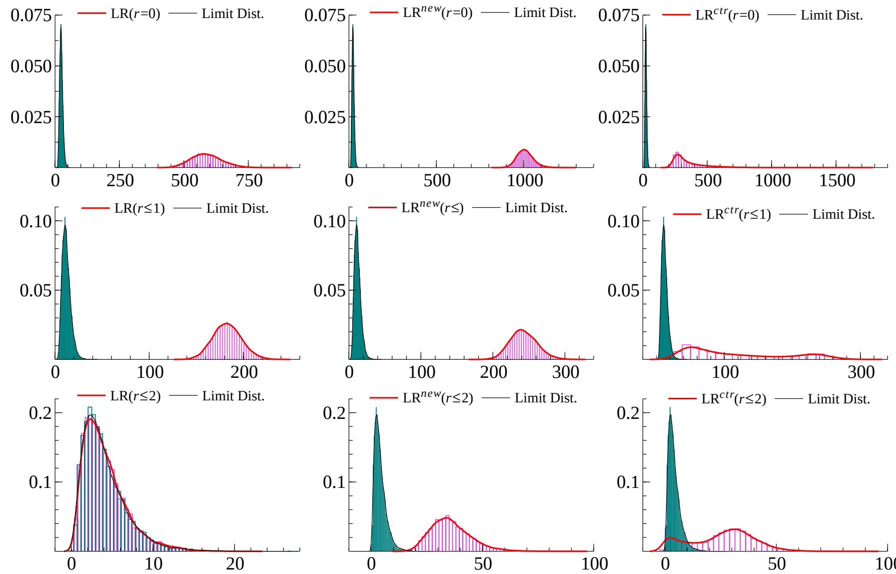

Figure 8: Distribution of cointegration test statistics at a sample of $T = 1,000$ observations for $p = 3$ and $r = 2$.

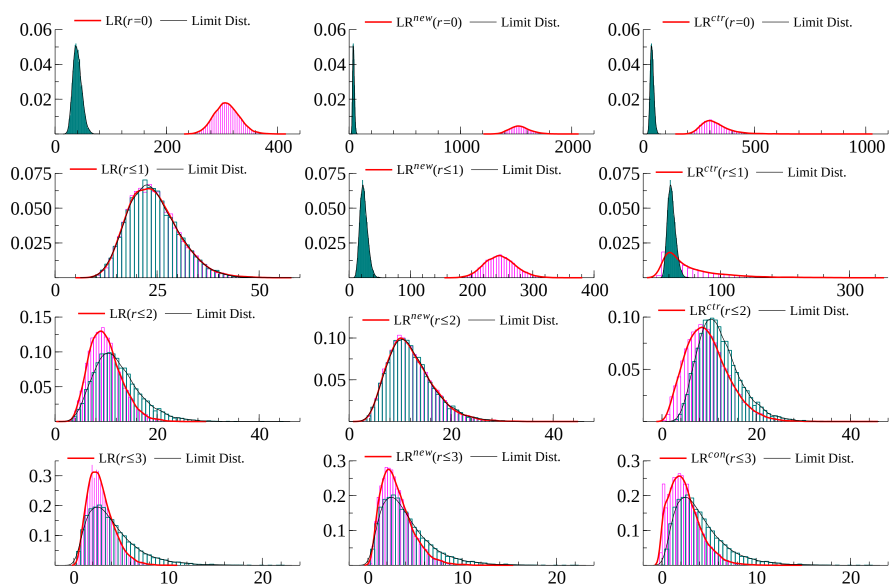

Figure 9: Distribution of cointegration test statistics at a sample of $T = 1,000$ observations for $p = 4$ and $r = 1$.

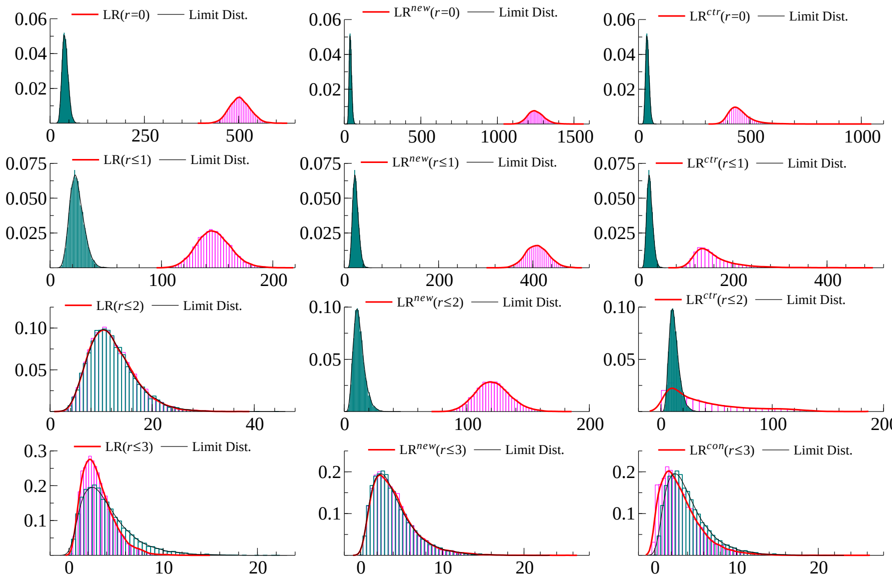

Figure 10: Distribution of cointegration test statistics at a sample of $T = 1,000$ observations for $p = 4$ and $r = 2$.

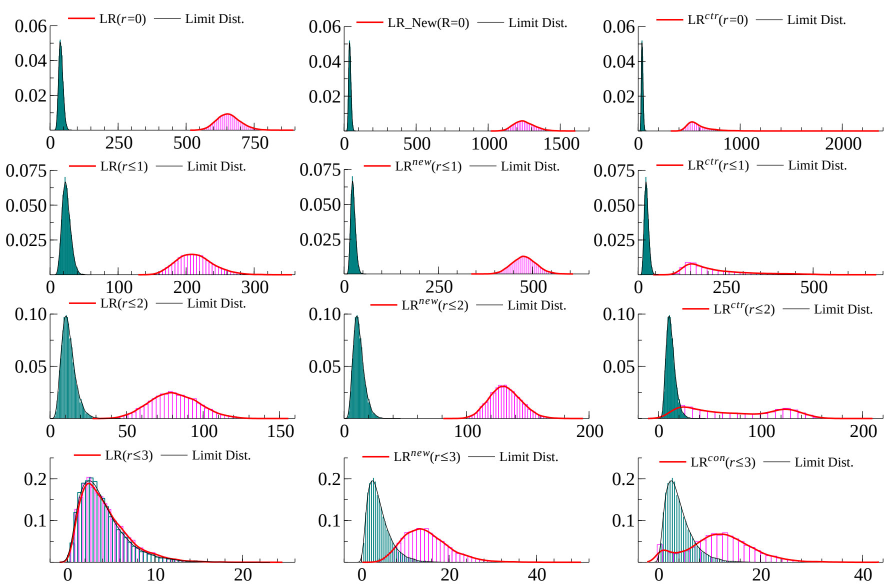

Figure 11: Distribution of cointegration test statistics at a sample of $T = 1,000$ observations for $p = 4$ and $r = 3$.

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